Simulation Study of Causality Change with Infinite Order Vector Autoregressive Processes

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Abstract — **The causality relationship between the time series is affected by the change in parameters of model. However, the change in parameters does not tell us the magnitude of causality change. In this study, we explore the measure of causality change between the time series and propose the test statistic whether there is any significance change in the causal relationship using frequency domain causality measure. To avoid the misspecification of the model, we employ infinite vector autoregressive models and the sieve approximation with increasing lag orders with sample size. The properties of the measure and test statistic are examined through the Monte Carlo simulation.**

Keyword: *Causality; autoregressive sieve estimation; structural change.*

1 Introduction

Economic time series often has structural change that incurred by some event such as policy change or financial crisis. When there is a structural change, the time series is characterized by different parameters of models for the sample periods before and after the change point. However it is not clear what kind of change in the causal relationship are carried by the change in the model parameters. This paper proposes a measure of change in the causal relationship and a statistical testing procedure for the significance of the change. The causality measure function is defined in the frequency domain as proposed in Geweke (1982) and Hosoya (1991). The frequency wise causality measure can be regarded as the decomposition of well known Granger causality measure that based on the variance ratio of the one step ahead prediction errors between two different models.

The inference procedure for the frequency wise causality measure proposed in the previous studies is only for the finite order vector autoregressive (VAR) model. However, such finite order VAR models are often insufficient to estimate the spectral densities of true generation process. In such case, the causality measure can not be estimated correctly so that it brings a wrong implication for the analysis. Instead of the finite order VAR model, we consider the VAR model with larger lag order which serves to avoid such difficulties. Lewis and Reinsel (1985) introduce an infinite order VAR model and establish the consistency and asymptotic normality for the estimated VAR coefficients. In this paper, the statistical inference for the causality measure function is introduced under infinite order VAR model. The parameters of model are estimated by ordinary least square (OLS) estimation. The causality measure functions are estimated as the nonlinear function of the estimated model parameters. The causality change can be defined as the difference of two causality measures for the periods before and after the change point. The asymptotic variance of the causality change which is required for the hypothesis testing is derived with the standard delta method for finite order VAR model. However, for the infinite order VAR model, we propose a bootstrap procedure to obtain the variance of the estimated change in the causal measure. The finite sample performance of the test with the bootstrap variance is investigated by Monte Carlo simulations.

Next section, we give a brief explanation and definitions for VAR model and the related causality measure. The asymptotic distributionof the change in the causality measures for the different sample periods is also given. The estimation procedure of the variance of the change in the causality measures

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is presented in section 3. We conduct a series of Monte Carlo experiments to see the properties of the test statistic in section 4. Section 5 gives some discussion.

2 Methodology

2.1 Causality measure for VAR Model

Let $Z(t) = [X(t)\,Y(t)]'$, we consider the following VAR model

$$A(L)Z(t) = \epsilon(t), \quad \epsilon(t) \sim N(0, \Omega), \tag{2.1}$$

where $A_p(L) = I - A_1L - A_2L^2 - \cdots - A_pL^p$ and $A(L) = \lim_{p\to\infty} A_p(L)$. It is assumed that the zeros of $\det(A(L))$ are inside of the unit circle. The moving average (MA) representation of (2.1) is

$$\begin{aligned} Z(t) &= A(L)^{-1}\epsilon(t) \\ &= \Psi(L)u(t), u(t) \sim N(0, I), \end{aligned}$$

where $\Psi(L) = A(L)^{-1}\Omega^{1/2}$ and $\Omega^{1/2}$ is the cholesky decomposition of Ω. Let parameter vector $\theta = \{\text{vec}(A_1)', \text{vec}(A_2)' \ldots, \text{vec}(A_p)', \text{vech}(\Omega)'\}'$, the causality measure $Y(t)$ to $X(t)$ at frequency λ is given by

$$M_{Y\to X}(\lambda; \theta) = \log\left\{1 + \frac{\|\Psi_{12}(e^{-i\lambda})\|}{\|\Psi_{11}(e^{-i\lambda})\|}\right\}. \tag{2.2}$$

Note that the causality measure function $M_{Y\to X}(\lambda;\theta)$ is the function of VAR parameters, θ.

2.2 VAR model with structural break

We assume that a data is generated by infinite order VAR model with structural break which described as follows. In this paper, it is assumed that a break point is known and defined as a constant fraction of sample size. Let T be the sample size and T_1 be the break point, $T_1 = [cT]$ where [x] means Gauss symbol. The VAR model with structural break is as follows

$$A^{(k)}(L)Z(t) = \epsilon(t), \quad \epsilon(t) \sim N(0, \Omega_k) \tag{2.3}$$

where $A_p^{(k)}(L) = I - A_{1,k}L - A_{2,k}L^2 - \cdots - A_{p,k}L^p$ and $A^{(k)}(L) = \lim_{p\to\infty} A_p^{(k)}(L)$ and $k = 1$ if $t < T_1$ and $k = 2$ if $t \geq T_1$. For the notational simplicity, we express the lag order of the model as p although which depends on k. We approximate (2.3) by VAR models with increasing lag order p which satisfies $p^3/T \to 0$ as

$$A_p^{(k)}(L)Z(t) = \epsilon^*(t), \quad \epsilon^*(t) \sim N(0, \Omega_k^*).$$

We define parameter vectors of VAR models before and after break as $\theta_k = \{\text{vec}(A_{1,k})', \text{vec}(A_{2,k})', \ldots, \text{vec}(A_{p,k})', \text{vech}(\Omega_k^*)'\}'$ and $k = 1, 2$.

2.3 Inference for the change in the causality measure

We introduce a test statistic for the causality change. The parameters are estimated by OLS and those are denoted by $\hat\theta_k = \{\text{vec}(\hat A_{1,k})', \text{vec}(\hat A_{2,k})', \ldots, \text{vec}(\hat A_{p,k})', \text{vech}(\hat\Omega_k^*)'\}'$. The causality measure functions are estimated by casting estimated parameters into (2.2). Since $\hat\theta_1$ and $\hat\theta_2$ are asymptotically independent so that $M_{Y\to X}(\lambda;\hat\theta_1)$ and $M_{Y\to X}(\lambda;\hat\theta_2)$ are also asymptotically independent. Let $m(\lambda;\theta_1,\theta_2) = M_{Y\to X}(\lambda;\theta_1) - M_{Y\to X}(\lambda;\theta_2)$. Assuming that true causality measure functions $M_{Y\to X}(\lambda;\theta_1)$ and $M_{Y\to X}(\lambda;\theta_2)$ are both not zero, the asymptotic distribution of $m(\lambda;\hat\theta_1,\hat\theta_2)$ is given by

$$\sqrt{\frac{T}{p}}m(\lambda;\hat\theta_1,\hat\theta_2) \xrightarrow{d} N\Big(\, m(\lambda;\theta_1,\theta_2),\ \mathcal{V}(\lambda,\theta)\,\Big)$$

where $\mathcal{V}(\lambda, \theta) = (1/c)V(\lambda, \theta_1) + (1/(1-c))V(\lambda, \theta_2)$ and $V(\lambda, \theta_1)$, $V(\lambda, \theta_2)$ are the asymptotic variance of $\sqrt{cT/p}\, M_{Y\to X}(\lambda; \hat{\theta}_1)$, $\sqrt{(1-c)T/p}\, M_{Y\to X}(\lambda; \hat{\theta}_2)$ respectively. Therefore the test statistic for null hypothesis $H_0 : m(\lambda; \theta_1, \theta_2) = 0$ is given by

$$m(\lambda; \hat{\theta}_1, \hat{\theta}_2)/\sqrt{v_1 + v_2} \xrightarrow{d} N(0, 1)$$

where $v_1 = p(cT)^{-1}V(\lambda, \hat{\theta}_1)$ and $v_2 = p((1-c)T)^{-1}V(\lambda, \hat{\theta}_2)$.

3 Estimation of the variance of the estimator

The asymptotic variance $\mathcal{V}(\lambda, \theta)$ is the complicated nonlinear function of the parameters. To estimate the asymptotic variance, we employ the bootstrap procedure. The bootstrap variance of estimator is calculated as the sample variance of repeated samples of the following procedure.

1. Bootstrap residuals $\hat{\epsilon}(t)_b$ are calculated using OLS residuals $\hat{\epsilon}(t)$ before break ($t < T_1$) and after break ($t \geq T_1$) respectively. In this step, OLS residuals are resampled as a vector.

2. Using $\hat{\epsilon}(t)_b$, the bootstrap data $Z(t)_b$ are generated as

$$\hat{A}_{p,k}(L)Z(t)_b = \hat{\epsilon}(t)_b,$$

   where $k = 1$ if $t < T_1$ and $k = 2$ if $t \geq T_1$.

3. Estimating parameters with $Z(t)_b$ by OLS, we get bootstrap samples of the parameter estimates which are denoted by $\hat{\theta}_{1,b}$ and $\hat{\theta}_{2,b}$.

4. Compute the causality measure functions $M(\lambda; \hat{\theta}_{1,b})$ and $M(\lambda; \hat{\theta}_{2,b})$.

With bootstrap variance of $M(\lambda; \hat{\theta}_1)$ and $M(\lambda; \hat{\theta}_2)$ which are denoted by $v_{1,B}$ and $v_{2,B}$, we construct a following test statistic,

$$m(\lambda; \hat{\theta}_1, \hat{\theta}_2)/\sqrt{v_{1,B} + v_{2,B}} \xrightarrow{d} N(0, 1).$$

4 Simulation

We demonstrate a test of causality changes for a infinite order VAR model with Gegenbauer polynomial as follows,

$$\begin{aligned} X(t) &= (0.3 + b(L))Y(t-1) + e_1(t) \\ Y(t) &= e_2(t) \end{aligned}$$

where $b(L) = \prod_{j=1}^{n}[\alpha(1 - 2\cos\lambda_j L - L^2)]$ and $[e_1(t)\ e_2(t)]' \sim N(0, I)$. We set $\lambda_j = 0.4 + 0.5(j/n)$. The break point is set to a half of sample size ($T_1 = 0.5T$) and T=2000, 4000, 8000. The left panel of Figure 1 shows the case of $n = 5$ and $\alpha = 0$ before break so that $M_{Y\to X}(\lambda; \theta_1)$ is constant at each frequency and $\alpha = 0.3$ after break. This case implies that $m(\lambda; \theta_1, \theta_2) = 0$ for any $\lambda \in [0.5\pi, \pi)$ as $n \to \infty$. It is noted that $m(\lambda; \theta_1, \theta_2)$ is not exactly equal to zero except for $\lambda = 0.5\pi, 0.6\pi, 0.7\pi$ and 0.9π since we set $n = 5$. The right panel of Figure 1 shows rejection rates for the causality change at each frequency. The nominal size for tests is 5% which is depicted by dashed line. The actual size is smaller than the nominal size at $\lambda \in [0.5\pi, \pi)$ when the sample size is $T = 2000$ and it gets close to the nominal size as the sample size becomes large. The power is high enough at low frequencies as we expected.

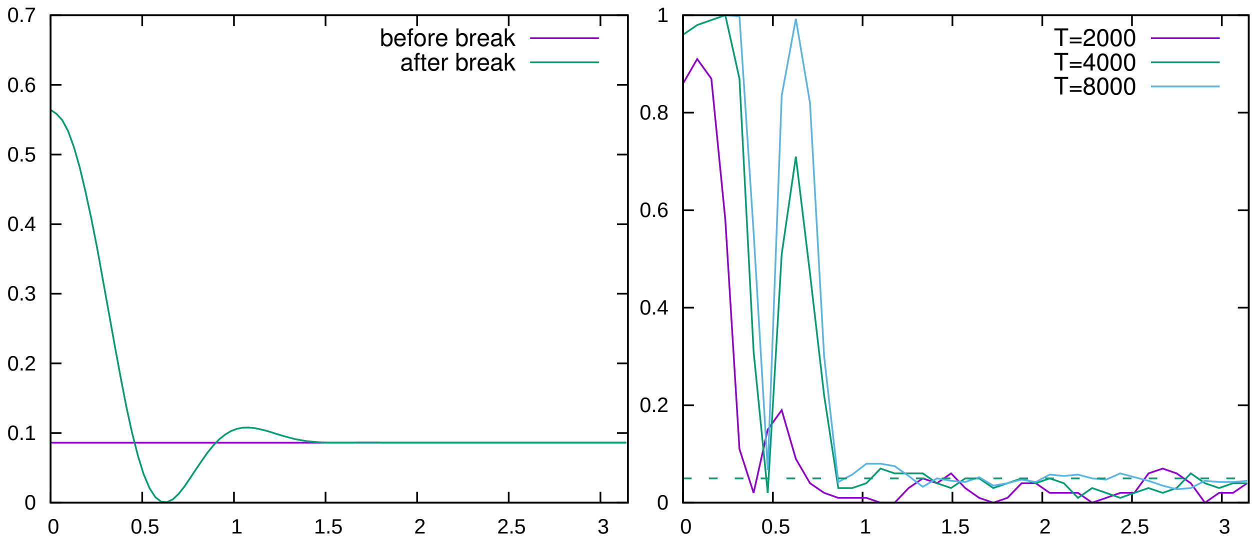

Figure 1: True causality measures (left panel) and Rejection rates for the causality change (right panel)

5 Discussions and summary

In this paper, a statistical inference for causality change is proposed with infinite order VAR model. The proposed method is an extension of Kinoshita and Oya (2014) which derive an inference for finite order VAR model. It is shown that the asymptotic distribution of the causality change is normal, however its asymptotic variance is complicated function of the parameters. A bootstrap procedure is proposed to estimate the asymptotic variance. Monte Carlo simulations shows that there is under bias in test size and it gets close to the nominal size as the sample size becomes large. Although we assume that the lag order is known in our simulation, it is unknown in empirical applications. We have several candidates for the lag selection such that a linear function of $T^{1/4}$, AIC and the general to specific approach proposed by Kuersteiner (2005). An inference with large lag order is asymptotically valid, however there may be a size distortion when sample size is finite or the true lag order is finite. It is necessary to try to find a well balanced lag selection procedure even for the infinite order VAR models.

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